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C^*-ALGEBRAS GENERATED BY FOURIER-STIELTJES TRANSFORMS(¹)

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Abstract. For G a locally compact group and $\hat{G}$ its dual, let $\mathscr{M}_d(\hat{G})$ be the C^*-algebra generated by the Fourier-Stieltjes transforms of the discrete measures on G. We show that the canonical trace on $\mathscr{M}_d(\hat{G})$ is faithful if and only if G is amenable as a discrete group. We further show that if G is nondiscrete and amenable as a discrete group, then the only measures in $\mathscr{M}_d(\hat{G})$ are the discrete measures, and also the sup and lim sup norms are identical on $\mathscr{M}_d(\hat{G})$. These results are extensions of classical theorems on almost periodic functions on locally compact abelian groups.

We present here some results dealing with nonabelian extensions of the theory of the von Neumann mean on the almost periodic functions. These results were previously announced in [3]. For G a locally compact group, let $\hat{G}$ be the dual of G. For $\pi \in \hat{G}$ and $\mu \in M(G)$, the measure algebra of G, let $\pi(\mu)$ be the Fourier-Stieltjes transform of μ at π. Let $\|\mu\|_\infty$ be $\sup\{\|\pi(\mu)\| : \pi \in \hat{G}\}$, and let $\mathscr{M}(\hat{G})$ be the C^*-completion of $M(G)$ relative to the norm $\|\cdot\|_\infty$. Let $\mathscr{M}_a(\hat{G})$, $\mathscr{M}_d(\hat{G})$ be the closures in $\mathscr{M}(\hat{G})$ of $L^1(G)$ (the space of measures absolutely continuous with respect to left Haar measure), $M_d(G)$ (the space of discrete measures) respectively. The algebra $\mathscr{M}_d(\hat{G})$ is a nonabelian analogue of the classical algebra of almost periodic functions.

In §1 we give some results dealing with the topology of $\hat{G}$ and also the spectrum of $\mathscr{M}(\hat{G})$, denoted by $\kappa\hat{G}$. In the abelian case this is the closure of the dual group of G in the spectrum of $M(G)$. We define the von Neumann trace on $\mathscr{M}(\hat{G})$ and derive some consequences.

In §2 we investigate the C^*-extension of the canonical projection which maps a measure to its discrete part. This makes possible a characterization of the null space of the trace, and a proof that $\kappa\hat{G}\backslash\hat{G}$ contains a homeomorphic copy of the reduced dual of G_d, the group G made discrete.

In §3 we show that the trace is faithful on $\mathscr{M}_d(\hat{G})$ if and only if G_d is amenable. We further show that if G is nondiscrete and G_d is amenable then the sup and lim sup norms are identical on $\mathscr{M}_d(\hat{G})$, and if $\mu \in \mathscr{M}_d(\hat{G})$ then $\mu \in M_d(G)$.

1. For G a locally compact group, we write $M(G)$ for the measure algebra of G, namely the set of finite regular Borel measures on G, $M_d(G)$ for the discrete

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measures in $M(G)$, and $L^1(G)$ for those measures in $M(G)$ which are absolutely continuous with respect to left (equivalently right) Haar measure. Then $M_d(G)$ is a closed subalgebra of $M(G)$ and $L^1(G)$ is a closed ideal. We define an involution on $M(G)$ by $\mu^*(E)=(\mu(E^{-1}))^-$ for each Borel set $E\subset G$, $\mu\in M(G)$.

DEFINITION 1.1. The dual of G, denoted by $\hat{G}$, is the set of equivalence classes of continuous irreducible unitary representations of G. As usual, however, we will take $\hat{G}$ to be a set of representations, one from each class; so if $\pi\in\hat{G}$, then π is a continuous homomorphism of G into the group of unitary operators on a Hilbert space $\mathscr{H}_\pi$.

DEFINITION 1.2. For $\mu\in M(G)$, $\pi\in\hat{G}$, let $\pi(\mu)=\int_G \pi(x)\,d\mu(x)$, a weak integral. Then $\pi(\mu)$ is a bounded operator on $\mathscr{H}_\pi$ with $\|\pi(\mu)\|\leqq\|\mu\|$, and $\pi(\mu^*)=\pi(\mu)^*$.

REMARK 1.3. Now π is an irreducible *-representation of $M(G)$ on $\mathscr{H}_\pi$ (see [1, p. 317]).

DEFINITION 1.4. For $\mu\in M(G)$ let $\|\mu\|_\infty=\sup\{\|\pi(\mu)\| : \pi\in\hat{G}\}$. Note that $\|\mu\|_\infty=0$ implies $\mu=0$ (see [1, p. 317]). Let $\mathscr{M}(\hat{G})$ be the completion of $(M(G), \|\cdot\|_\infty)$, then $\mathscr{M}(\hat{G})$ is a C^*-algebra, and $M(G)$ is identified with a dense subalgebra. Denote the spectrum of $\mathscr{M}(\hat{G})$ by $\kappa\hat{G}$. Recall the spectrum of a C^*-algebra is the set of (equivalence classes of) irreducible *-representations furnished with the Zariski topology induced by the kernels of these representations, the primitive ideals (see [1, p. 60]). Further let $\mathscr{M}_a(\hat{G})$ be the closure of $L^1(G)$ in $\mathscr{M}(\hat{G})$, thus $\mathscr{M}_a(\hat{G})$ is a closed ideal. Finally let $\mathscr{M}_d(\hat{G})$ be the closure of $M_d(G)$ in $\mathscr{M}(\hat{G})$, so $\mathscr{M}_d(\hat{G})$ is a C^*-subalgebra.

DEFINITION 1.5. For $S\subset\kappa\hat{G}$ let $\mathscr{N}(S)=\{\phi\in\mathscr{M}(\hat{G}) : \pi(\phi)=0 \text{ for all } \pi\in S\}$. Then $\mathscr{N}(S)$ is a closed ideal in $\mathscr{M}(\hat{G})$. Let $\mathscr{M}(S)=\mathscr{M}(\hat{G})/\mathscr{N}(S)$ be the quotient C^*-algebra (see [1, p. 17]).

The topology on $\kappa\hat{G}$ may be defined as follows: for subsets S, T of $\kappa\hat{G}$, $S\subset\overline{T}$ if and only if $\mathscr{N}(S)\supset\mathscr{N}(T)$. It is known (see [1, p. 65]) that $\kappa\hat{G}$ is a compact topological space, but not necessarily T_0. Further $\hat{G}$ is identified with a dense open subset of $\kappa\hat{G}$, since $\hat{G}=\{\pi\in\kappa\hat{G} : \pi(\mathscr{M}_a(\hat{G}))\neq 0\}$ (see [1, p. 61]).

The left regular representation of G is defined to be the left translation operation on $L^2(G)$ (L^2-space of left Haar measure), that is, the map $x\mapsto\lambda(x)$ where $\lambda(x)f(y)=f(x^{-1}y)$ for $x, y\in G, f\in L^2(G)$. Then the reduced dual of G, denoted by $\hat{G}_r$, is the set of $\{\pi\in\hat{G} : \pi \text{ is weakly contained in } \lambda\}$ (see [1, p. 317]). For $\mu\in M(G)$, $\lambda(\mu)$ is given by $\lambda(\mu)f=\mu*f, f\in L^2(G)$, then let

$$\|\mu\|_r = \|\lambda(\mu)\| = \sup\{\|\mu*f\|_2 : f\in L^2(G), \|f\|_2\leqq 1\}.$$

An equivalent definition of the reduced dual is $\{\pi\in\hat{G} : \|\pi(\mu)\|\leqq\|\mu\|_r \text{ for all } \mu\in M(G)\}$. Then $\|\mu\|_r=\sup\{\|\pi(\mu)\| : \pi\in\hat{G}_r\}$. Observe for $\mu\in M(G)$ that $\|\mu\|_r=0$ implies $\mu=0$ so that $\mathscr{N}(\hat{G}_r)\cap M(G)=\{0\}$.

We now define a trace on $M(G)$ and extend it to $\mathscr{M}(\hat{G})$.

DEFINITION 1.6. For $\mu\in M(G)$ let $\operatorname{Tr}(\mu)=\mu\{e\}$ (e is the identity of G). Then

$|\mathrm{Tr}\,(\mu)| \leqq \|\mu\|_r \leqq \|\mu\|_\infty$ (see [2]), so Tr extends uniquely to a bounded linear functional on $\mathscr{M}(\hat{G})$.

The functional Tr has the following properties, where $\mu, \nu \in M(G)$ and $\mu\nu$ denotes the convolution of μ and ν:

(1) $\mathrm{Tr}\,(\mu\nu) = \mathrm{Tr}\,(\nu\mu) = \sum_{x\in G} \mu\{x\}\nu\{x^{-1}\}$,

(2) $\mathrm{Tr}\,(\mu^*\mu) = \sum_{x\in G} |\mu\{x\}|^2 \geqq 0$.

Since measures of the form $\mu^*\mu$ are dense in the positive cone of $\mathscr{M}(\hat{G})$, we see that Tr is actually a finite trace on $\mathscr{M}(\hat{G})$, that is, $\mathrm{Tr}\,(\phi\psi) = \mathrm{Tr}\,(\psi\phi)$ and $\mathrm{Tr}\,(\phi^*\phi) \geqq 0$ for $\phi, \psi \in \mathscr{M}(\hat{G})$.

DEFINITION 1.7. Let $\mathrm{null}\,\mathrm{Tr} = \{\phi \in \mathscr{M}(\hat{G}) : \mathrm{Tr}\,(\phi^*\phi) = 0\}$. Then by the Cauchy-Schwarz inequality and other obvious reasons null Tr is a closed two-sided ideal in $\mathscr{M}(\hat{G})$.

Observe that $\mathrm{Tr}\,(\phi) = 0$ for $\phi \in \mathrm{null}\,\mathrm{Tr}$ and further that $\mathscr{N}(\hat{G}_r) \subset \mathrm{null}\,\mathrm{Tr}$, for if $\phi \in \mathscr{N}(\hat{G}_r)$ then $0 \leqq \mathrm{Tr}\,(\phi^*\phi) \leqq \|\phi^*\phi\|_r = 0$. If G is nondiscrete then $\mathscr{M}_a(\hat{G}) \subset \mathrm{null}\,\mathrm{Tr}$.

2. Denote the locally compact group G made discrete by G_d. Then $\hat{G}_d$ is the dual of G_d and is also the spectrum of $\mathscr{M}(\hat{G}_d) = \mathscr{M}_a(\hat{G}_d) = \mathscr{M}_d(\hat{G}_d)$. Each $\pi \in \hat{G}$ gives an irreducible unitary representation of G_d, thus $\hat{G}$ is identified with a subset of $\hat{G}_d$. We denote the closure of $\hat{G}$ in $\hat{G}_d$ by $\hat{G}_{dc}$. It does not seem to be known whether $\hat{G}_{dc} = \hat{G}_d$ in general, although it is true in the abelian case.

Further denote the reduced dual of G_d by $\hat{G}_{dr}$, the set of $\pi \in \hat{G}_d$ which are weakly contained in the left regular representation of G_d on $l^2(G_d)$. Observe that $M_d(G)$ can be identified with $M(G_d)$, and $\mathscr{M}_d(\hat{G}) \cong \mathscr{M}(\hat{G}_{dc})$. The trace Tr is also defined on $\mathscr{M}(\hat{G}_d)$.

THEOREM 2.1. *There is a unique C^*-homomorphism of P $\mathscr{M}(\hat{G})$ onto $\mathscr{M}(\hat{G}_{dr})$ such that, for $\mu \in M(G)$, $P\mu$ is the discrete part of μ, and* kernel $P \supset \mathscr{N}(\hat{G}_r)$ *(see Definition* 1.5).

Proof. Let $\mu \in M(G)$ be written as $\mu = \mu_d + \mu_c$, where μ_d is discrete and μ_c is continuous (zero on countable sets). We now show that $\|\mu_d\|_{dr} \leqq \|\mu\|_r$ where

$$\|\mu_d\|_{dr} = \sup\{\|\pi(\mu_d)\| : \pi \in \hat{G}_{dr}\} = \sup\{\|\mu_d * f\|_2 : f \in l^2(G_d),\ \|f\|_2 \leqq 1\};$$

so we will show that $\|\mu_d * f\|_2 \leqq \|\mu\|_r \|f\|_2$ for $f \in l^2(G_d)$. We may assume that μ_d and f are finitely supported. So there are finite sets $\{x_i\}_{i=1}^n$, $\{y_j\}_{j=1}^m$ in G, with $y_1 = e$, and complex numbers $\{a_i\}$ and $\{b_j\}$ such that $f(x_i) = a_i$, $i = 1, \ldots, n$, and $f(x) = 0$ otherwise; and $\mu_d = \sum_j b_j\,\delta_{y_j}$ (δ_y is the unit mass at y). Let $E = \{y_j x_i : 1 \leqq i \leqq n, 1 \leqq j \leqq m\}$, and let U be a compact neighborhood of e such that $zU \cap z'U = \varnothing$ whenever $z, z' \in E$ and $z \neq z'$. Another smallness condition will be placed on U later. Now let $g = \sum_{i=1}^n a_i \chi_{x_i U} \in L^2(G)$ (where χ_E is the characteristic function of the set E), then

$$\|g\|_2 = (m(U))^{1/2}\Big(\sum_i |a_i|^2\Big)^{1/2} = (m(U))^{1/2}\|f\|_2$$

(m is left Haar measure on G).

A simple computation shows that

$$(\mu_d+\mu_c)*g(v) = \sum_{z\in E}\Big(\sum_{y_jx_i=z} a_ib_j\Big)\chi_{zU}(v)+\sum_{i=1}^{n} a_i\mu_c(vU^{-1}x_i^{-1}) \quad (\text{for } v\in G).$$

The first term on the right side is 0 off EU. Now

$$\|\mu*g\|_2^2 = \int_{G\setminus EU}|\mu_c*g|^2\,dm+\sum_{z\in E}\int_{zU}\left|\sum_{y_jx_i=z}a_ib_j+\sum_{i=1}^{n}a_i\mu_c(vU^{-1}x_i^{-1})\right|^2 dm(v)$$

$$\geqq m(U)\left\{\left(\sum_{z\in E}\left|\sum_{y_jx_i=z}a_ib_j\right|^2\right)^{1/2}-\sup_{v\in EU}\left|\sum_{i=1}^{n}a_i\mu_c(vU^{-1}x_i^{-1})\right|\right\}^2$$

(Cauchy-Schwarz),

provided that the second term in $\{\cdot\}$ is smaller than the first term, which is easily computed to be $\|\mu_d*f\|_2$. Now we make the second term arbitrarily small by suitable choice of U. Observe that for $v\in zU$ some $z\in E$ we have that

$$|\mu_c(vU^{-1}x_i^{-1})| \leqq |\mu_c|(zUU^{-1}x_i^{-1}), \quad 1\leqq i\leqq n.$$

But $|\mu_c|$ is continuous, so for $\varepsilon>0$ and $E=\{z_k\}_{k=1}^N$, there exists for each i,j, with $1\leqq i,j\leqq N$ a neighborhood V_{ij} of e such that $|\mu_c|(z_iz_j^{-1}V_{ij})<\varepsilon$. Now let U further satisfy the condition $UU^{-1}\subset\bigcap_{i,j=1}^N(z_j^{-1}V_{ij}z_j)$, then $|\mu_c|(z_iUU^{-1}z_j^{-1})<\varepsilon$ for all i,j. Recall $\{x_i\}_{i=1}^n\subset E$, since $y_1=e$. Now for $\varepsilon>0$, we can pick U so that $|\mu_c|(zUU^{-1}x_i^{-1}) <\varepsilon/\sum_{j=1}^n|a_j|$ for each $z\in E$, $1\leqq i\leqq n$. Thus $m(U)(\|\mu_d*f\|_2-\varepsilon)^2\leqq\|\mu*g\|_2^2 \leqq\|\mu\|_r^2\,\|g\|_2^2=m(U)\|\mu\|_r^2\,\|f\|_2^2$. The map $\mu\mapsto\mu_d$ thus extends to a C^*-homomorphism of $\mathscr{M}(\hat{G})$ into $\mathscr{M}(\hat{G}_{dr})$ with dense range, hence onto [1, p. 18.]. □

COROLLARY 2.2. *For* $\phi\in\mathscr{M}(\hat{G})$, Tr $(P\phi)=$ Tr (ϕ).

Proof. For $\mu\in M(G)$, Tr $(P\mu)=\mu_d\{e\}=\mu\{e\}=$ Tr (μ). □

THEOREM 2.3. *The trace* Tr *is faithful on* $\mathscr{M}(\hat{G}_{dr})$, *that is*, $\phi\in\mathscr{M}(\hat{G}_{dr})$ *with* $\phi\geqq 0$ *and* Tr $(\phi)=0$ *implies* $\phi=0$.

Proof. We use the method of von Neumann [5, p. 484]. Observe that any bounded operator T on $l^2(G_d)$ which commutes with all right translations is of the form $Tf(x)=\sum_{y\in G}F_T(xy^{-1})f(y)$, where F_T is a square-summable function on G_d (this is not a sufficient condition to be a bounded operator). Clearly $\mathscr{M}(\hat{G}_{dr})$ is a C^*-algebra of such operators, and Tr $(\phi)=F_\phi(e)$. Now if $\phi\geqq 0$ then $\phi=\psi^*\psi$ for some $\psi\in\mathscr{M}(\hat{G}_{dr})$ and Tr $(\phi)=\sum_{y\in G}|F_\psi(y)|^2$, so Tr $(\phi)=0$ implies $\psi=0$. □

COROLLARY 2.4. *The kernel of* $P=$ null Tr (*as ideals in* $\mathscr{M}(\hat{G})$).

THEOREM 2.5. *For* $\mu\in M_d(G)$, $\|\mu\|_{dr}\leqq\|\mu\|_r\leqq\|\mu\|_\infty$; *and thus* $\hat{G}_{dc}\supset\hat{G}_{dr}$.

Proof. The norm inequalities are from Theorem 2.1 and show that $\mathscr{N}(\hat{G}_{dr})\supset\mathscr{N}(\hat{G}_{dc})$. □

Now we observe that if G is nondiscrete then $\kappa\hat{G}\setminus\hat{G}$ contains a homeomorphic copy of $\hat{G}_{dr}$.

THEOREM 2.6. *If G is nondiscrete and $\pi \in \hat{G}_{dr}$ then $\pi \circ P$ is an irreducible representation of $\mathscr{M}(\hat{G})$ and $\pi \circ P \in \kappa\hat{G}\backslash\hat{G}$. Further the map $\pi \to \pi \circ P$ is a homeomorphism.*

Proof. In fact $\mathscr{M}_a(\hat{G}) \subset$ kernel P, and P maps $\mathscr{M}(\hat{G})$ onto $\mathscr{M}(\hat{G}_{dr})$. □

DEFINITION 2.7. Let G be nondiscrete and $S \subset \hat{G}$. Then define a seminorm on $\mathscr{M}(\hat{G})$, called S-lim sup, to be the quotient norm of $\mathscr{M}(S)/\mathscr{M}_a(S)$. Recall $\mathscr{M}(S) = \mathscr{M}(\hat{G})/\mathscr{N}(S)$ and $\mathscr{M}_a(S) = \mathscr{M}_a(\hat{G})/(\mathscr{N}(S) \cap \mathscr{M}_a(\hat{G}))$.

The algebra $\mathscr{M}(S)/\mathscr{M}_a(S)$ can also be viewed as the quotient algebra $\mathscr{M}(\hat{G})/(\mathscr{M}_a(\hat{G}) + \mathscr{N}(S))$. Note that $\mathscr{M}_a(\hat{G}) + \mathscr{N}(S)$ is the closed two-sided ideal generated by $\mathscr{M}_a(\hat{G})$ and $\mathscr{N}(S)$; see [1, p. 18].

PROPOSITION 2.8. *Let $S, T \subset \hat{G}$ with $S \subset \overline{T}$, then*

$$S\text{-lim sup}(\phi) \leqq T\text{-lim sup}(\phi) \quad \textit{for each } \phi \in \mathscr{M}(\hat{G}).$$

Proof. Since $S \subset \overline{T}$ we have $\mathscr{N}(S) \supset \mathscr{N}(T)$ so there is a canonical *-homomorphism j of $\mathscr{M}(T)$ onto $\mathscr{M}(S)$. The kernel of $\mathscr{M}(T) \to \mathscr{M}(S)/\mathscr{M}_a(S)$ is $j^{-1}\mathscr{M}_a(S)$ which contains $\mathscr{M}_a(T)$. □

REMARK 2.9. If G is compact or abelian then the $\hat{G}$-lim sup is identical to $\limsup_{\pi \to \infty} \|\pi(\phi)\| = \inf_K \{\sup \|\pi(\phi)\|, \pi \notin K\}$, K a compact subset of $\hat{G}$, for $\phi \in \mathscr{M}(\hat{G})$. In the general locally compact case we are only able to show that $\limsup_{\pi \to \infty} \|\pi(\phi)\| \leqq \hat{G}$-lim sup (ϕ) for $\phi \in \mathscr{M}(\hat{G})$.

THEOREM 2.10. *If G is nondiscrete, then* $\hat{G}$-lim sup $(\phi) \geqq \hat{G}_r$-lim sup $(\phi) \geqq \|P\phi\|_{dr}$ *for each $\phi \in \mathscr{M}(\hat{G})$.*

Proof. The following chain of inclusions of ideals holds: $\mathscr{M}_a(\hat{G}) \subset \mathscr{M}_a(\hat{G}) + \mathscr{N}(\hat{G}_r) \subset$ kernel P. □

3. DEFINITION 3.1. A locally compact group G is said to be amenable if there exists a left invariant mean on the space of bounded continuous functions (see [4]). Equivalent characterizations are that $\hat{G} = \hat{G}_r$, or that the representation $G \to \{1\}$ is in $\hat{G}_r$ (see [6]).

Under the assumption that G_d is amenable, we can prove direct extensions of certain abelian-case theorems.

PROPOSITION 3.2. *$\hat{G}_{dc} = \hat{G}_{dr}$ if and only if G_d is amenable.*

Proof. If G_d is amenable then $\hat{G}_d \supset \hat{G}_{dc} \supset \hat{G}_{dr} = \hat{G}_d$. If G_d is not amenable, then $G_d \to \{1\}$ is not contained in $\hat{G}_{dr}$ but it is contained in $\hat{G}_{dc}$. □

THEOREM 3.3. *The trace* Tr *is faithful on $\mathscr{M}_d(\hat{G})$ if and only if G_d is amenable.*

Proof. Observe that $\mathscr{M}_d(\hat{G}) \cong \mathscr{M}(\hat{G}_{dc})$, and (kernel P) $\cap\, \mathscr{M}_d(\hat{G}) \cong \mathscr{N}(\hat{G}_{dr})/\mathscr{N}(\hat{G}_{dc})$. But kernel $P =$ null Tr, so Tr is faithful on $\mathscr{M}_d(\hat{G})$ if and only if (null Tr) $\cap\, \mathscr{M}_d(\hat{G}) = \{0\}$ if and only if $\mathscr{N}(\hat{G}_{dr}) = \mathscr{N}(\hat{G}_{dc})$. □

If G_d is amenable then so is G, thus $\hat{G} = \hat{G}_r$.

PROPOSITION 3.4. *If G_d is amenable then $P|\mathcal{M}_d(\hat{G})$ is an isomorphism onto $\mathcal{M}(\hat{G}_d)$.*

THEOREM 3.5. *If G_d is amenable, $\phi \in \mathcal{M}_d(\hat{G})$, then $\hat{G}$-lim sup $(\phi) = \|\phi\|_\infty$. Further if $\mu \in M(G)$, then $\|\mu\|_\infty \geqq \hat{G}$-lim sup $(\mu) \geqq \|P\mu\|_\infty = \hat{G}$-lim sup $(P\mu)$.*

Proof. Let μ_d be the discrete part of $\mu \in M(G)$, then by Theorem 2.10 $\|\mu\|_\infty \geqq \hat{G}$-lim sup $(\mu) \geqq \|P\mu\|_{dr} = \|\mu_d\|_{dr} = \|\mu_d\|_{dc} = \|\mu_d\|_\infty$. Now extend to $\mathcal{M}_d(\hat{G})$. □

THEOREM 3.6. *If G_d is amenable, then $\mathcal{M}(\hat{G}) = \mathcal{M}_c(\hat{G}) \oplus \mathcal{M}_d(\hat{G})$, where $\mathcal{M}_c(\hat{G})$ is the closure in $\mathcal{M}(\hat{G})$ of the set of continuous measures in $M(G)$. Further* null Tr $= \mathcal{M}_c(\hat{G})$.

Proof. We saw above that $P|\mathcal{M}_d(\hat{G})$ is an isomorphism onto $\mathcal{M}(\hat{G}_d)$, so that we can define a unique C^*-projection P' on $\mathcal{M}(\hat{G})$ whose range is $\mathcal{M}_d(\hat{G})$, such that $P'(\mu) = \mu_d$, where μ_d is the discrete part of $\mu \in M(G)$. Observe that null Tr = kernel P = kernel $P' = (I-P')\mathcal{M}(\hat{G})$ (where I is the identity operator). It now suffices to show that $(I-P')\mathcal{M}(\hat{G}) = \mathcal{M}_c(\hat{G})$. If $\mu \in M(G)$ is continuous then $(I-P')\mu = \mu - P'\mu = \mu$; thus $(I-P')\phi = \phi$ for any $\phi \in \mathcal{M}_c(\hat{G})$. Conversely let $\phi \in$ kernel P', then there exists a sequence of measures $\{\mu_n\} \subset M(G)$ such that $\|\mu_n - \phi\|_\infty \xrightarrow{n} 0$. Then

$$\|(I-P')\mu_n - \phi\|_\infty = \|(I-P')(\mu_n - \phi)\|_\infty \leqq \|\mu_n - \phi\|_\infty \xrightarrow{n} 0,$$

and each $(I-P')\mu_n$ is a continuous measure. □

COROLLARY 3.7. *If G_d is amenable and $\mu \in M(G)$ and $\mu \in \mathcal{M}_d(\hat{G})$ then $\mu \in M_d(G)$.*

Examples. Any solvable locally compact group G has a solvable discrete group G_d, which is thus amenable [4, p. 9]. If G_d contains the free group on two generators then G_d is not amenable. For example if G contains $SO(3)$ (the rotation group on R^3) or $SU(2)$ (the group of unitary matrices on C^2 of determinant 1) then G_d is not amenable [4, p. 9].

Also if G is not amenable then G_d is a priori not amenable. The noncompact semisimple connected Lie groups are examples of nonamenable groups.

If G is locally finite, that is, any finitely generated subgroup is finite, then G_d is amenable (see, for example, [4, p. 30]). A large class of examples of locally finite groups is constructed as follows.

THEOREM 3.8. *Let $\{G_\lambda : \lambda \in \Lambda$ (index set)$\}$ be a set of finite groups such that* $\sup_\lambda$ (order $G_\lambda) < \infty$, *then the complete direct product $G = \prod_{\lambda \in \Lambda} G_\lambda$ is a locally finite compact group and thus G_d is amenable.*

Proof. We may assume that each $G_\lambda = H$, some fixed finite group (since any subgroup of a locally finite group is locally finite). Write $g \in G$ as $g = (g_\lambda)_{\lambda \in \Lambda}$, $g_\lambda \in H$, and let $H = \{h_1, \ldots, h_m\}$. Each $g \in G$ defines a finite partition $\{E(g)_i\}_{i=1}^m$ of Λ, where $E(g)_i = \{\lambda : g_\lambda = h_i\}$. Let $\{g^{(1)}, \ldots, g^{(n)}\} = S$, a finite set in G, then for any $g \in$ Gp (S), the group generated by S, the function $\lambda \mapsto g_\lambda$ is constant on each set $E(g^{(1)})_{i_1} \cap \cdots \cap E(g^{(n)})_{i_n}$, $1 \leqq i_j \leqq m$. These sets form a finite partition of Λ with at most

m^n elements. But the function $\lambda \mapsto g_\lambda$ is finitely valued, thus order $(\mathrm{Gp}(S)) \leq m^{n+1}$. □

This proof was pointed out to us by our colleague L. Pitt.

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